Frame Fields: Anisotropic and Non-Orthogonal Cross Fields
Additional material

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This additional material provides a formal algebraic definition of frame fields as generalizations of N-symmetry (a.k.a. N-RoSy) fields and a rigorous statement of the basic theory developed in the paper.

1 Algebra of frame fields

Let $\mathcal{S}$ be a smooth orientable surface embedded in $\mathbb{R}^3$ and let p be a point on $\mathcal{S}$. We define n_p to be the surface normal of $\mathcal{S}$ at p, $\mathbf{T}_p\mathcal{S}$ the tangent plane at p and $\mathbf{T}\mathcal{S}$ the tangent bundle of $\mathcal{S}$. A chart for $\mathcal{S}$ is a pair (U, ϕ) where U is a subset of $\mathcal{S}$ and $\phi : U \to \mathbb{R}^2$ is a homeomorphism of U onto an open subset of $\mathbb{R}^2$; an *atlas* for $\mathcal{S}$ is a collection of charts $\{(U_\alpha, \phi_\alpha) | \alpha \in A\}$ such that $\cup_{\alpha\in A} U_\alpha = \mathcal{S}$.

A vector field $\mathcal{F} : \mathcal{S} \longrightarrow \mathbf{T}\mathcal{S}$ maps each point p to a vector lying on $\mathbf{T}_p\mathcal{S}$. Given an atlas for $\mathcal{S}$, as above, $\mathcal{F}$ is smooth at p if and only if $\mathcal{F} \circ \phi_\alpha^{-1}$ is smooth for all U_α containing p.

In the following, we generalize the concept of (smooth) vector fields by considering different equivalence classes of vectors on tangent planes. A simple example is the *direction field*, in which we factor out the length of the vectors. For $\mathbf{u}, \mathbf{v}$ vectors, we define the equivalence class:

$$\mathbf{v} \sim_1 \mathbf{u} \;\;\Leftrightarrow\;\; \mathbf{v} = a\mathbf{u} \;\text{ for some scalar } a > 0. \tag{1}$$

If we consider the quotient space of each tangent plane with respect to this equivalence relation, and the related quotient tangent bundle $\mathbf{T}\mathcal{S}/\sim_1$, a field $\mathcal{F}_D : \mathcal{S} \longrightarrow \mathbf{T}\mathcal{S}/\sim_1$ maps each point on the surface to a direction. It is customary to take unit-length vectors as representatives of their equivalence classes, so that a direction field can be regarded as a vector field where the length of all vectors is 1. This is equivalent to identifying the quotient space of the tangent plane at p to the unit circle centered at p. Note that if we take a vector field $\mathcal{F}$ and we define its corresponding directional field $\mathcal{F}/\sim_1$ by mapping each vector in the image of $\mathcal{F}$ to its representative direction, then $\mathcal{F}/\sim_1$ is undefined at points where $\mathcal{F}$ vanishes. Vanishing points of vector fields, as well as isolated points where directional fields are undefined, are called *singularities*.

1.1 Rotational symmetry fields

Let $\mathcal{V} \simeq \mathbb{R}^2$ be a two-dimensional Euclidean vector space; let $\mathcal{C} \subset \mathcal{V}$ be the set of all unit-length vectors of $\mathcal{V}$. Let $\Theta_{\frac{2\pi}{n}} : \mathcal{V} \to \mathcal{V}$ be the endomorphism that rotates a vector by the angle of $\frac{2\pi}{n}$; the restriction of $\Theta_{\frac{2\pi}{n}}$ to $\mathcal{C}$ is also an endomorphism. For an integer $k \geq 0$, let us define the concatenation of k instances of $\Theta_{\frac{2\pi}{n}}$ as

$$\Theta^k_{\frac{2\pi}{n}} = \Theta_{\frac{2\pi}{n}} \circ \ldots \circ \Theta_{\frac{2\pi}{n}} = \Theta_{\frac{2k\pi}{n}}. \tag{2}$$

We have $\Theta^n_{\frac{2\pi}{n}} = Id$, hence there exist just n distinct endomorphisms $\Theta^k_{\frac{2\pi}{n}}$, with $k = 0, \ldots, n-1$. We define the following equivalence relation on $\mathcal{C}$ for a given n:

$$\mathbf{u} \sim_n \mathbf{v} \;\;\Leftrightarrow\;\; \mathbf{u} = \Theta^k_{\frac{2\pi}{n}}(\mathbf{v}) \text{ for some } k \geq 0. \tag{3}$$

The quotient space $\mathcal{C}/\sim_n$ is called an n-RoSy space (the name RoSy is borrowed from [Palacios and Zhang 2007]). An element of such a space, i.e., an n-RoSy, can be represented as the collection of the n unit-length vectors of $\mathcal{C}$ that belong to the same equivalence class. Alternatively, any vector of $\mathcal{C}$ can be taken as a representative of its equivalence class in the n-RoSy space.

Let $\mathcal{S}$ be a smooth surface and p a point on $\mathcal{S}$, as before. If we take $\mathbf{T}_p\mathcal{S}$ as the vector space $\mathcal{V}$ and the unit circle on $\mathbf{T}_p\mathcal{S}$ centered at p as $\mathcal{C}$, then we can define the n-RoSy space on the tangent plane of p, denoted as $\mathcal{RS}^n_p = \mathbf{T}_p\mathcal{S}/\sim_n$. Analogously, we define the quotient tangent bundle, i.e., the collection of all n-RoSy spaces for all points of $\mathcal{S}$, as

$$\mathcal{RS}^n_{\mathcal{S}} = \mathbf{T}\mathcal{S}/\sim_n. \tag{4}$$

An *n-RoSy field* on $\mathcal{S}$ is a field $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{RS}^n_{\mathcal{S}}$. RoSy fields have been studied by several authors in the literature. The works of Palacios and Zhang [2007] and Ray et al. [2008] provide several results on RoSy fields, among which the definition of smoothness, curvature and turning numbers that characterize singularities.

Note that n-RoSy's identify vectors of any length that can be mapped onto each other by an integer multiple of rotation $\Theta_{\frac{2\pi}{n}}$. Therefore, they abstract both size (as direction fields do) and rotations for fixed *period jumps*. Note also that direction fields as defined above correspond to 1-RoSy fields.

A 4-RoSy field is commonly called a *cross field*. In the following, we generalize cross fields to obtain fields that incorporate the concepts of *scale*, *anisotropy* and *skewness*.

1.2 Frame fields

From now on, we restrict ourselves to $n = 4$, while introducing several generalizations to cross fields. For convenience, we rename the 4-RoSy tangent bundle $\mathcal{RS}^4_{\mathcal{S}}$ of a surface $\mathcal{S}$ as the *cross space* of $\mathcal{S}$, and we denote it by $\mathcal{CS}_{\mathcal{S}}$.

Let $\mathcal{V} \simeq \mathbb{R}^2$ be again a two-dimensional Euclidean vector space. Let us consider the endomorphism on $\mathcal{V} \times \mathcal{V}$ defined as $R(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, -\mathbf{u})$, and its concatenation defined as $R^k = R \circ \ldots \circ R$ where R appears k times in the concatenation. We have $R^4 = Id$, thus there exist only 4 distinct functions $R^0 = Id, R^1, R^2$ and R^3. We define the following equivalence relation on $\mathcal{V} \times \mathcal{V}$:

$$(\mathbf{u}, \mathbf{v}) \sim_R (\mathbf{u}', \mathbf{v}') \;\;\Leftrightarrow\;\; (\mathbf{u}, \mathbf{v}) = R^k(\mathbf{u}', \mathbf{v}') \quad \text{for some } k \in \{0, 1, 2, 3\}. \tag{5}$$

The quotient space $(\mathcal{V} \times \mathcal{V})/\sim_R$ is called the *frame space* of $\mathcal{V}$. A *frame*, i.e., an element of $(\mathcal{V} \times \mathcal{V})/\sim_R$, can be represented as a cyclically ordered set of four vectors $\langle \mathbf{u}, \mathbf{v}, -\mathbf{u}, -\mathbf{v} \rangle$, where the angle brackets denote cyclic order. The four representatives in $\mathcal{V} \times \mathcal{V}$ of an element of $(\mathcal{V} \times \mathcal{V})/\sim_R$ are the pairs of consecutive elements in such a cyclic order, i.e.: $(\mathbf{u}, \mathbf{v})$, $(\mathbf{v}, -\mathbf{u})$, $(-\mathbf{u}, -\mathbf{v})$ and $(-\mathbf{v}, \mathbf{u})$.

Similarly to what we did before, we can define the *frame space* $\mathcal{FS}_{\mathcal{S}}$ of $\mathcal{S}$ as the quotient space $\mathbf{T}\mathcal{S}^2/\sim_R$ of the two-dimensional tangent bundle of $\mathcal{S}$, and then define a *frame field* as a function

$$\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{FS}_{\mathcal{S}}.$$

In what follows, we will only consider *non-degenerate, right-handed* frames, i.e., the equivalence classes of pairs $(\mathbf{u}, \mathbf{v})$ of linearly independent vectors that form (counterclockwise) an angle θ, $0 < \theta < \pi$. We will thus require that all frames in the image of a frame field are non-degenerate, right-handed.

A cross field can be regarded as a special case of frame field, in which the representatives of all frames are orthonormal pairs of vectors. Algebraically, this can be obtained by a canonical projection of the tangent bundle into the two-dimensional tangent bundle, and extending such projections to their respective quotient spaces, as follows:

$$\begin{array}{ll} E : \mathbf{T}\mathcal{S} \longrightarrow \mathbf{T}\mathcal{S}^2, & \mathbf{u} \mapsto (\mathbf{u}, \mathbf{u}^{\perp}) \\ E_R : \mathcal{CS}_{\mathcal{S}} \longrightarrow \mathcal{FS}_{\mathcal{S}}, & [\mathbf{u}] \mapsto \langle \mathbf{u}, \mathbf{u}^{\perp}, -\mathbf{u}, -\mathbf{u}^{\perp} \rangle \end{array} \qquad (6)$$

where $\mathbf{u}^{\perp} = \Theta_{\frac{\pi}{2}}(\mathbf{u})$ is the vector orthogonal to $\mathbf{u}$ obtained by rotating $\mathbf{u}$ by the angle of $\frac{\pi}{2}$, and $[\mathbf{u}]$ denotes the equivalence class of vector $\mathbf{u}$ in the cross space. Note that E_R is well defined, i.e., it gives the same cyclic order of vectors, no matter what vector $\mathbf{u}$ is used to represent a given cross. Given a cross field $\mathcal{X}$, its frame field version is trivially $E_R \circ \mathcal{X}$.

Conversely, a frame can be regarded as the scaled and sheared version of a cross. We first extend linear maps in the plane to linear maps in the frame space. Let $\mathcal{V} \simeq \mathbb{R}^2$ be a two-dimensional Euclidean vector space, and $\mathbf{W} : \mathcal{V} \longrightarrow \mathcal{V}$ a non-singular linear map (i.e., $\det \mathbf{W} \neq 0$ if $\mathbf{W}$ is represented by a matrix on a basis for $\mathcal{V}$). We naturally extend the map $\mathbf{W}$ to a *frame linear map* as follows:

$$\begin{array}{rcl} \mathbf{W}_R : (\mathcal{V} \times \mathcal{V})/{\sim_R} & \longrightarrow & (\mathcal{V} \times \mathcal{V})/{\sim_R}, \qquad (7) \\ \langle \mathbf{v}, \mathbf{w}, -\mathbf{v}, -\mathbf{w} \rangle & \mapsto & \langle \mathbf{W}(\mathbf{v}), \mathbf{W}(\mathbf{w}), \mathbf{W}(-\mathbf{v}), \mathbf{W}(-\mathbf{w}) \rangle . \end{array}$$

Note that $\mathbf{W}_R$ is well defined since $\mathbf{W}$ is linear and non-singular, thus, for any $\mathbf{v}$ we have $\mathbf{W}(-\mathbf{v}) = -\mathbf{W}(\mathbf{v})$.

Now let us consider a unit-length vector $\mathbf{u} \in \mathbf{T}_p\mathcal{S}$, where p is a point on $\mathcal{S}$. Let us take the cross $[\mathbf{u}] \in \mathcal{CS}_{\mathcal{S}}$ and a linear map $\mathbf{W}^p : \mathbf{T}_p\mathcal{S} \longrightarrow \mathbf{T}_p\mathcal{S}$. If we deform each vector of $[\mathbf{u}]$ through $\mathbf{W}^p$, we obtain a frame, i.e., $\mathbf{W}^p_R \circ E_R([\mathbf{u}])$ is a frame in $\mathcal{FS}_{\mathcal{S}}$, which deforms $[\mathbf{u}]$ through $\mathbf{W}^p_R$. Next we show that there is a canonical way to represent a frame as a pair formed of a cross and a symmetric deformation.

Lemma 1.1. ***Canonical decomposition**.*
Let $f_{\mathbf{v},\mathbf{w}} = \langle \mathbf{v}, \mathbf{w}, -\mathbf{v}, -\mathbf{w} \rangle \in \mathcal{FS}_{\mathcal{S}}$ be a non-degenerate, right-handed frame. There exists a unique cross field $[\mathbf{u}] \in \mathcal{CS}_{\mathcal{S}}$ and a unique symmetric positive definite *(SPD) linear map $\mathbf{W}$ such that $f_{\mathbf{v},\mathbf{w}} = \mathbf{W}_R(E_R([\mathbf{u}])$.*

Proof: Set a local Euclidean reference system $(\mathbf{x}, \mathbf{y})$ in the vector space spanned by $f_{\mathbf{v},\mathbf{w}}$ (tangent plane). Then $\mathbf{v}$ and $\mathbf{w}$ have coordinates $(\mathbf{v}_x, \mathbf{v}_y)$ and $(\mathbf{w}_x, \mathbf{w}_y)$ in this reference system, respectively. Since $f_{\mathbf{v},\mathbf{w}}$ is non-degenerate and right-handed, the matrix

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_x & \mathbf{w}_x \\ \mathbf{v}_y & \mathbf{w}_y \end{pmatrix}$$

has full rank and positive determinant. Therefore, $\mathbf{V}$ admits a unique polar decomposition $\mathbf{V} = \mathbf{UP}$ where $\mathbf{U}$ is a rotation matrix and $\mathbf{P}$ is a symmetric positive definite matrix. We may rewrite the polar decomposition as $\mathbf{WU} = \mathbf{V}$ where $\mathbf{W} = \mathbf{UPU}^T$ is also SPD. With abuse of notation, let us identify each vector with the column of its two coordinates. Let $\mathbf{u}$ be the unit-length vector corresponding to the first column of $\mathbf{U}$, i.e., $\mathbf{U} = [\mathbf{u}, \mathbf{u}^{\perp}]$ Let us build the 4×2 matrices $[\mathbf{u}, \mathbf{u}^{\perp}, -\mathbf{u}, -\mathbf{u}^{\perp}]$ and $[\mathbf{v}, \mathbf{w}, -\mathbf{v}, -\mathbf{w}]$, then we have

$$\mathbf{W}\,[\mathbf{u}, \mathbf{u}^{\perp}, -\mathbf{u}, -\mathbf{u}^{\perp}] \;=\; [\mathbf{v}, \mathbf{w}, -\mathbf{v}, -\mathbf{w}]. \qquad (8)$$

Any cyclic permutation of the four columns of the matrix built from $\mathbf{u}$ returns a corresponding cyclic permutation of the matrix built from $\mathbf{v}, \mathbf{w}$, thus we can conclude that $\mathbf{W}_R(E_R([\mathbf{u}]) = f_{\mathbf{v},\mathbf{w}}$. □

Next we extend the canonical decomposition to frame fields. In order to do this, we must first define linear maps acting on the tangent bundle of the surface $\mathcal{S}$. Let p be a point on $\mathcal{S}$; a linear map $\mathbf{W}_p : \mathbf{T}_p\mathcal{S} \longrightarrow \mathbf{T}_p\mathcal{S}$ is a linear function that associates to each vector $\mathbf{v}$ on the tangent plane $\mathbf{T}_p\mathcal{S}$ another vector lying on the same tangent plane. Let $\{(U_\alpha, \phi_\alpha) | \alpha \in A\}$ be an atlas for $\mathcal{S}$, let ϕ_α be defined at p, and let $\mathbf{J}_\alpha$ be the Jacobian of ϕ_α. Then we can represent $\mathbf{W}_p$ through the following commutative digram:

$$\begin{array}{ccc} \mathbf{T}_p\mathcal{S} & \xrightarrow{\mathbf{W}_p} & \mathbf{T}_p\mathcal{S} \\ \mathbf{J}_\alpha \big\downarrow & & \big\uparrow \mathbf{J}_\alpha^{-1} \\ \mathbb{R}^2 & \xrightarrow{\bar{\mathbf{W}}_p} & \mathbb{R}^2 \end{array}$$

where $\bar{\mathbf{W}}_p$ is a (uniquely defined) linear map in the Euclidean plane, expressed as a 2×2 matrix. Now let us define the tensor field

$$\mathcal{W} : \mathcal{S} \longrightarrow \mathcal{L}_{\mathcal{S}}, \qquad \mathcal{W}(p) = \mathbf{W}_p$$

where $\mathcal{L}_{\mathcal{S}}$ is the space of linear maps on the tangent bundle of $\mathcal{S}$. We can define a map

$$\omega_\alpha : \phi_\alpha(U_\alpha) \subset \mathbb{R}^2 \longrightarrow \mathbf{M}_{2,2}, \qquad \phi_\alpha(p) \mapsto \bar{\mathbf{W}}_p$$

where $\mathbf{M}_{2,2}$ is the space of 2×2 matrices. We say that $\mathcal{W}$ is smooth if and only if all ω_α are smooth according to the Frobenius norm on $\mathbf{M}_{2,2}$. We can extend the tensor field $\mathcal{W}$ to the frame space in a canonical way as follows:

$$\mathcal{W}_R : \mathcal{FS}_{\mathcal{S}} \longrightarrow \mathcal{FS}_{\mathcal{S}},$$

if $\mathbf{v}, \mathbf{w} \in \mathbf{T}_p\mathcal{S}$ then

$$\langle \mathbf{v}, \mathbf{w}, -\mathbf{v}, -\mathbf{w} \rangle \overset{\mathcal{W}_R}{\mapsto} \langle \mathbf{W}_p(\mathbf{v}), \mathbf{W}_p(\mathbf{w}), \mathbf{W}_p(-\mathbf{v}), \mathbf{W}_p(-\mathbf{w}) \rangle.$$

Finally, given a cross field $\mathcal{X}$ and a smooth tensor field $\mathcal{W}$ as above, we say that the frame field defined as $\mathcal{F}(p) = \mathcal{W}_R(E_R(\mathcal{X}(p)))$ is smooth if and only if both $\mathcal{X}$ and $\mathcal{W}$ are smooth. The proof of the following proposition readily follows:

Proposition 1.2. *Let $\mathcal{F}$ be a (non-degenerate, right-handed) frame field on $\mathcal{S}$ and for each $p \in \mathcal{S}$ let $(\mathbf{X}_p, \mathbf{W}_p)$ be the canonical decomposition of $\mathcal{F}(p)$ as defined in Lemma 1.1. Let $\mathcal{X}$ be the frame field obtained by collecting all $\mathbf{X}_p$'s, and $\mathcal{W}$ be the tensor field obtained by collecting all $\mathbf{W}_p$'s for all $p \in \mathcal{S}$. Then $\mathcal{F}$ is smooth if and only if both $\mathcal{X}$ and $\mathcal{W}$ are smooth.*

In summary, a frame field can be decomposed into a cross field and an SPD tensor field, and the smoothness of a frame field can be defined in terms of the smoothness of these two fields.

References

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